Mathematical Framework for Interpreting Pair Angular Correlations in a Two-Source Model

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Abstract

The mathematical description of two-particle pair measurments is reviewed, in general and for the specific case of azimuthal angular correlations. The conditional multiplicity observable is also examined and compared to the correlation observable. A two-source "jet/flow" model is exhibited, which expressly allows for the possibility that a jet source could respect the collision reaction plane. The pair correlations expected within the model are calculated in detail and discussed.

Contents

1 General Pair Observables

1.1 Differential Pair Multiplicity

The simplest and most basic observable for particle pairs is simply to count them; or, more exactly, to count all "pair outcomes" that occur in our event sample. Here we define a pair outcome, in fullest generality, as the production of a particle of type A into a bit of phase space $d^3\mathbf{p}_A$ around a momentum vector $\mathbf{p}_A$ in coincidence with a particle of type B into $d^3\mathbf{p}_B$ around $\mathbf{p}_B$. We denote the total count of such pairs over some event sample as d^6N^{AB}. If we call the total number of events in the sample N_{evt}, then the basic observable is the *differential pair multiplicity per event*

$$\frac{1}{N_{evt}}\frac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B} \tag{1}$$

The differential pair multiplicity is a function over six particle phase space dimensions, and contains all observable information on AB–pair production in the event sample[1].

Generally the labels A and B need not refer to just one unique type of particle. Each could also indicate a group or class of particles, such as all charged particles or all positive particles, for example. This, in turn, means that in general the classes A and B could be completely different, or *non-overlapping* sets; they could be *identical* sets; or they could be *partially overlapping* sets. In this document we will deal only with the non-overlapping and identical cases, and leave the partially-overlapping case to be worked out by the interested reader as required.

[1] This quantity is sometimes loosely called a "pair cross section"; and while this is technically incorrect it is often forgivable, since a multiplicity per event and a semi-inclusive cross section are identical up to a multiplicative constant.

1.2 Full Correlation Functions

The simplest case of pair production is that the production of each memeber of a pair is unaffected by the presence of the other on any event. In the non-overlapping case, this means specifically that the number and spectrum of A particles is independent of the number of and spectrum of B particles, and vice versa. If this is true then we say that the production of A and B are *uncorrelated*; the pair differential multiplicity per event will then factorize[2], and the factors will be the singles multiplicities per event:

$$\frac{1}{N_{evt}} \frac{d^6 N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B} = \frac{1}{N_{evt}} \frac{d^3 N^A}{d^3\mathbf{p}_A} \times \frac{1}{N_{evt}} \frac{d^3 N^B}{d^3\mathbf{p}_B} \tag{2}$$

If we integrate the pair multiplicity in the uncorrelated, non-overlapping case, it is obvious from looking at the RHS of Eq. 2 that the result must be simply the product of the average multiplicites per event for each A and B.

We can quantify the extent to which A and B production are correlated by measuring the degree to which Eq. 2 is violated. Several different such measures can be imagined; the traditional approach is to use the ratio, and so we define the *full correlation function* as

$$C(\mathbf{p}_A, \mathbf{p}_B) \equiv \frac{\dfrac{1}{N_{evt}} \dfrac{d^6 N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B}}{\left(\dfrac{1}{N_{evt}} \dfrac{d^3 N^A}{d^3\mathbf{p}_A}\right) \left(\dfrac{1}{N_{evt}} \dfrac{d^3 N^B}{d^3\mathbf{p}_B}\right)} \tag{3}$$

The function $C(\mathbf{p}_A, \mathbf{p}_B)$ is a real-valued, non-negative function over six momentum phase space dimensions. It will be equal to 1 everywhere if A and B production are completely uncorrelated (as defined above). In general the full correlation function can have arbitrarily high or low (non-negative) values and is not constrained by any simple sum rule; its value can be below 1 everywhere, above 1 everywhere, or any combination in between [*That's what makes it so much fun. – Ed.*]

1.3 Special Care For The Identical-Particle Case

Uncorrelated production in the identical-particle case, where A and B denote the same type of particle, takes a little more care since the average number of pairs per event is *not* trivially equal to the square of the average number of singles per event. The analog of Eq. 2 that we can write for the identical-particle case starts as a proportionality

$$\frac{1}{N_{evt}} \frac{d^6 N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B} \propto \left(\frac{1}{N_{evt}} \frac{d^3 N^A}{d^3\mathbf{p}_A}\right) \left(\frac{1}{N_{evt}} \frac{d^3 N^B}{d^3\mathbf{p}_B}\right) \tag{4}$$

[2]Technically, for this factorization to hold rigorously the spectrum of A particles on each event must also be independent of the number of A particles, in addition to the already-stated conditions — and the same, of course, must also hold true for B. This point rarely becomes important because it is already assumed in many models, but it should be kept in the back of one's mind.

where A and B now refer to the same class of particles, but the momenta $\mathbf{p}_A$ and $\mathbf{p}_B$ are distinct variables.

An event with n particles of chosen type will have $n(n-1)$ same-type pairs. So, if we integrate the LHS of Eq. 4 over all phase space, the result must be the average number of pairs per event $\overline{n(n-1)}$. The RHS of Eq. 4, meanwhile, clearly integrates to $(\overline{n})^2$. So in the uncorrelated identical-particle case the full analog of Eq. 2 is

$$\frac{1}{N_{evt}}\frac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B} = \frac{\overline{n(n-1)}}{(\overline{n})^2}\left(\frac{1}{N_{evt}}\frac{d^3N^A}{d^3\mathbf{p}_A}\right)\left(\frac{1}{N_{evt}}\frac{d^3N^B}{d^3\mathbf{p}_B}\right) \tag{5}$$

Though formally correct, in virtually all practical cases Eq. 5 has an immediate simplification. If the number of particles per event n is Poisson distributed[3], then it will be true that $\overline{n(n-1)} = (\overline{n})^2$ and Eq. 5 reduces to Eq. 2. In this case the form of the full correlation function in Eq. 3 can be used for the identical and non-overlapping cases equally well; in this document we will follow this path and use Eq. 3 for both cases.

1.4 Reduced Correlation Functions

Since the full correlation function is unwieldy as a function over six dimensions, it is more practical to work with a *reduced correlation function.* We define the reduced correlation function by integrating — separately — the numerator and denominator in Eq. 3 and then taking the ratio of the integrals. The integral can be over any volume of six-dimensional phase space, which we denote generically as Γ; typically this volume will be defined by some parameter(s), which we generically denote as α and the dependence as $\Gamma(\alpha)$. The reduced correlation function is defined for choice of Γ, and so is effectively a function of the parameter(s) α

$$C_{\text{Reduced}}(\alpha) \equiv \frac{\displaystyle\int_{\Gamma(\alpha)} d^3\mathbf{p}_A\, d^3\mathbf{p}_B\ \frac{1}{N_{evt}}\frac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B}}{\displaystyle\int_{\Gamma(\alpha)} d^3\mathbf{p}_A\, d^3\mathbf{p}_B \left(\frac{1}{N_{evt}}\frac{d^3N^A}{d^3\mathbf{p}_A}\right)\left(\frac{1}{N_{evt}}\frac{d^3N^B}{d^3\mathbf{p}_B}\right)} \tag{6}$$

Many different kinds of reduced correlation functions, each with its own $\Gamma(\alpha)$, can be defined to access different kinds of physics. For example, in HBT correlations the phase-space volume Γ might select all pairs with a particular value of pair invariant momentum difference q_{inv}; this would constitute the parameter which defines Γ, and so the reduced correlation function would become $C(q_{inv})$. Additionally we might limit Γ to include pairs within a certain pair p_T bin, and the bin definition becomes another parameter $C(q_{inv}, \text{pair}\, p_T\, \text{bin})$.

[3]In all realistic mechanisms where particle production is uncorrelated the multiplicity distributions will also be Poisson, so this assumption is generally quite safe. Some would go so far as to argue that any mechanism of particle production that qualifies for the description "uncorrelated" *necessarily* results in a Poisson multiplicity distribution. We will not open such a debate here, but just say that for all practical cases Eq. 2 is the *definition* of "uncorrelated" in both the identical and non-overlapping cases, and then use Eq. 3 universally.

Note that, whenever the full correlation function has a constant value over all phase space, then *all* reduced correlation functions will have that same value, for all values of their parameters. In particular, in the case of fully uncorrelated production all reduced correlation functions will equal 1 for all parameters.

One particularly simple case is when the integration volume Γ factorizes as the product of a volume Γ_A over $\mathbf{p}_A$ and Γ_B over $\mathbf{p}_B$. We refer to this as a "factorized" or "bin-reduced" correlation function:

$$C_{\text{Factorized}} = \frac{\int_{\Gamma_A} d^3\mathbf{p}_A \int_{\Gamma_B} d^3\mathbf{p}_B \, \frac{1}{N_{evt}} \frac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B}}{\left(\int_{\Gamma_A} d^3\mathbf{p}_A \, \frac{1}{N_{evt}} \frac{d^3N^A}{d^3\mathbf{p}_A}\right) \left(\int_{\Gamma_B} d^3\mathbf{p}_B \frac{1}{N_{evt}} \frac{d^3N^B}{d^3\mathbf{p}_B}\right)} \tag{7}$$

We can see that $C_{\text{Factorized}}$ has a very simple interpretation. The numerator is the average number of pairs, per event, with $\mathbf{p}_A$ in Γ_A and $\mathbf{p}_B$ in Γ_B. The denominator is the product of the average number of singles for these two conditions. So the correlation function is just the ratio between an average number of pairs per event and the product of two average numbers of singles per event:

$$C_{\text{Factorized}} = \frac{\left(\frac{N(\mathbf{p}_A \,\text{in}\, \Gamma_A \cap \mathbf{p}_B \,\text{in}\, \Gamma_B)}{N_{evt}}\right)}{\left(\frac{N(\mathbf{p}_A \,\text{in}\, \Gamma_A)}{N_{evt}}\right) \left(\frac{N(\mathbf{p}_B \,\text{in}\, \Gamma_B)}{N_{evt}}\right)} \tag{8}$$

The illustration in Eq.'s 7 and 8 is for a single, fixed selection of Γ_A and Γ_B; in such a case $C_{\text{Factorized}}$ is just a constant (and the pedantically inclined might argue against calling it a "function"). Of course, the factorized correlation function could also be a function of some parameter(s) $C_{\text{Factorized}}(\alpha)$, as long as $\Gamma_A(\alpha)$ and $\Gamma_B(\alpha)$ are functions of the same parameter(s) and the factorization $\Gamma(\alpha) = \Gamma_A(\alpha) \times \Gamma_B(\alpha)$ holds.

1.5 Residual Correlations

The simplest case of a non-trivial correlation in pair production is *residual correlations.* We say that the production of A and B particles show only a residual correlation *if* we can divide the full event sample into sub-samples *such that* the production of A and B particles is *uncorrelated* within each sub-sample. Suppose the sub-samples can be identified by the value of some parameter Q, which could be either continuous or discrete. Then the singles and pairs distributions for events in sub-sample Q are all functions of Q, and are related by

$$\frac{1}{N_{evt}} \frac{d^6N^{AB}(Q)}{d^3\mathbf{p}_A d^3\mathbf{p}_B} = \frac{1}{N_{evt}} \frac{d^3N^A(Q)}{d^3\mathbf{p}_A} \; \frac{1}{N_{evt}} \frac{d^3N^B(Q)}{d^3\mathbf{p}_B} \tag{9}$$

Over the full event sample the parameter Q has some probability distibution $P(Q)$ which we take to be normalized to a total sum/integral of 1. Then it is trivial to write the full singles and pairs distributions over the full event sample; in the case that Q is discrete

$$\frac{1}{N_{evt}}\frac{d^3N^A}{d^3\mathbf{p}_A} = \sum_Q P(Q)\,\frac{1}{N_{evt}}\frac{d^3N^A(Q)}{d^3\mathbf{p}_A} \tag{10}$$

$$\frac{1}{N_{evt}}\frac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B} = \sum_Q P(Q)\,\frac{1}{N_{evt}}\frac{d^6N^{AB}(Q)}{d^3\mathbf{p}_A d^3\mathbf{p}_B} \tag{11}$$

or in the case that Q is continuous

$$\frac{1}{N_{evt}}\frac{d^3N^A}{d^3\mathbf{p}_A} = \int dQ\,P(Q)\,\frac{1}{N_{evt}}\frac{d^3N^A(Q)}{d^3\mathbf{p}_A} \tag{12}$$

$$\frac{1}{N_{evt}}\frac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B} = \int dQ\,P(Q)\,\frac{1}{N_{evt}}\frac{d^6N^{AB}(Q)}{d^3\mathbf{p}_A d^3\mathbf{p}_B} \tag{13}$$

with the obvious corresponding definition for the singles distribution for B, of course. To obtain the full or reduced correlation functions these full-sample distributions are then inserted into Eq. 3 or Eq. 6, respectively, and the resulting correlation functions are said to show the evidence of the residual correlations only.

These definitions of residual correlations may seem a bit abstract at this point, but a simple example should show that the notion is quite intuitive.

1.6 Centrality Spread: A Worked Example

A residual correlation that is nearly always present in analyzing heavy-ion collision stems simply from the fact that all event samples will cover some range in collision centrality. To see how this simple fact can lead to a residual correlation, let's consider the very simplest imaginable model of particle production: suppose that every A+A collision has a well-defined number of interacting nucleons N_{Part}; and further suppose that the multiplicity for each type of particle on any given event is proportional to N_{part} for that event, but are otherwise uncorrelated, and that the shapes of all the spectra are independent of N_{Part}. Then we can write the singles and pairs spectra for events with fixed N_{Part} as

$$\frac{1}{N_{evt}}\frac{d^3N^A(N_{Part})}{d^3\mathbf{p}_A} = a\,N_{Part}\,F_A(\mathbf{p}_A)$$

$$\frac{1}{N_{evt}}\frac{d^3N^B(N_{Part})}{d^3\mathbf{p}_B} = b\,N_{Part}\,F_B(\mathbf{p}_B) \tag{14}$$

$$\frac{1}{N_{evt}}\frac{d^6N^{AB}(N_{Part})}{d^3\mathbf{p}_A d^3\mathbf{p}_B} = ab\,(N_{Part})^2\,F_A(\mathbf{p}_A)F_B(\mathbf{p}_B) \tag{15}$$

where a and b are constants and F_A and F_B are normalized functions describing the spectral shapes. We can easily see how this model fits the definition of a residual correlation: the sub-samples are events of fixed N_{Part}, with N_{Part} as the defining parameter (obviously); and for events in a sub-sample A and B production are explicitly uncorrelated. The equations 14 and 15 are the incarnation of the general equation 9.

If we now say that N_{Part} has some (normalized) probability distribution $P(N_{Part})$, then we can calculate the particle distributons for the full event sample using Eq.'s 10 and 11 and write the full correlation function

$$\begin{aligned} C(\mathbf{p}_A, \mathbf{p}_B) &= \frac{\sum_{N_{Part}} P(N_{Part}) ab (N_{Part})^2 F_A(\mathbf{p}_A) F_B(\mathbf{p}_B)}{\left(\sum_{N_{Part}} P(N_{Part}) a N_{Part} F_A(\mathbf{p}_A)\right)\left(\sum_{N_{Part}} P(N_{Part}) b N_{Part} F_B(\mathbf{p}_B)\right)} \\ &= \frac{\sum_{N_{Part}} P(N_{Part})\,(N_{Part})^2}{\left(\sum_{N_{Part}} P(N_{Part})\,N_{Part}\right)^2} \\ &= 1 + \left(\frac{\sigma(N_{Part})}{\overline{N_{Part}}}\right)^2 \end{aligned} \tag{16}$$

where $\overline{N_{Part}}$ is the mean of the N_{Part} distribution for the full event sample, and $\sigma(N_{Part})$ is its standard deviation. We can see immediately that in this simplest imaginable model the full correlation function, and so all reduced correlation functions, will have a constant value everywhere which is larger than 1.

It is worth a moment to appreciate how large these finite-centrality-spread effects might be. In this simple example, suppose our full event sample is a minimum-bias or peripheral sample. If we say — very crudely – that the N_{Part} distribution for such a sample is flat from $N_{Part} = 0$ out to some maximum N_{Part}, then it follows immediately that the full correlation function will have a constant value of 4/3. On the other hand, for any narrow sample of central events the N_{Part} distribution will be fairly sharply peaked, with a much lower (sigma/mean), and the correlation function might be expected to differ from 1 by less than a percent. So, any correlation analysis in heavy-ion collisions which investigates a physics effect that would produce correlation functions different from 1 by amounts of order 1 or less will need some way to deal with the unavoidable presence of "uninteresting" correlations from other effects, such as centrality spread.

1.7 Subtleties of Normalizing Correlation Functions

The moral of the above story is that correlation functions don't always go to 1 even where you might think they should, *ie* where "there's no physics going on", if your event sample spans a significant range in centrality — and in the real world all event samples will span some range in centrality. So any correlation analysis in heavy-ion physics, even one with ideal, perfect measurements, has to cope with the fact that the correlation functions as defined in Eq.'s 3 and 6 can – and most likely will – show the effects of both the "interesting" correlation that you might want to study and the (usually) "uninteresting" effects of a finite centrality spread.

Traditionally there are a number of approaches to obviate these "uninteresting" correlation effects. One possibility is to systematically divide the full event sample into smaller samples each with a smaller centrality range and study how the correlation changes with the sample size. Note that this approach will illuminate the true, ideal correlation function only if the finite-spread correlation functions are themselves perfect objects, *ie* fully corrected for all instrumental and acceptance effects to produce the ideal quantity defined in Eq. 6. Since absolute normalizations can be very cumbersome to determine experimentally, this is not typically the first approach taken if an easier one will avail.

A simpler and more common approach is to presume that the effects of finite centrality spread will affect a correlation function only as an overall multiplicative constant. After this, you can in some cases choose the overall normalization for a correlation function to force its value to 1 at some point in phase/parameter space where you "know" it should be 1 — that is, at a point where you know the correlation effects of the physics you are interested in should be absent. A good example of this is the approach commonly taken in HBT correlation analyses, where the correlation functions are normalized to equal 1 at high values of pair relative momentum, where identical-particle effects on correlations should be vanishingly small.

One practical advantage of this approach is that since its overall normalization is determined separately, the correlation function only needs to be measured up to within some multiplicative constant. This greatly eases the work involved in an analysis, since the single-particle and pair acceptances now do *not* have to be determined absolutely. In fact, the only acceptance effect that remains relevant at all is the shape of the ratio between the pair acceptance and the product of the singles acceptances; and this ratio will typically be very flat over all pair phase space except for very close-angle pairs.

The simplest approach of all is to simply measure a correlation function up to an unknown constant, and just declare its normalization to be arbitrary. Such an "arbitrary units" correlation function is not the ideal object of Eq. 6, but it can still have great utility. For analyses where the true, ideal correlation function is not expected to differ much from 1, it is sometimes worthwhile for plotting purposes to normalize a correlation function so that its average or typical value is at or near 1. For example, if the correlation function's shape is quite flat it can be normalized so that its total integral is the same as integrating a constant of 1 over the chosen range of the parameter(s) α. As a slight variation, the correlation function can be normalized to have the integrals over the parameter(s) α of the numerator and denominator in Eq. 6 be equal; this choice is sometimes referred to as "area normalization". It should be appreciated, though, that in general the integral of a reduced correlation function over its parameter(s) α is not usually a meaningful quantity – though it can be in some cases.

Happily, at least for this document, the relative-angle correlation *is* one case in which the integral of the correlation function has a simple meaning, and this sum rule can be used as another means of normalizing angular correlation functions. See Sections 2.1 and 2.2 for discussion and details.

1.8 Conditional Multiplicity

In theory of probability, when two outcomes A and B are possible it is common to ask, "How likely is it that B will occur given that A has occurred?" In other words, what is the probability of B conditioned on the occurrence of A? Such a *conditional probability* is noted $P(B|A)$ and is found[4] to be equal to

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \tag{17}$$

where $P(A \cap B)$ is the probability of both A and B occurring, and $P(A)$ is the probability of A occurring without reference to B.

A closely analogous question in the analysis of pair production is, "On events with an A particle in $d^3\mathbf{p}_A$ around $\mathbf{p}_A$, what is the average multiplicity of B particles into $d^3\mathbf{p}_B$ around $\mathbf{p}_B$?" We denote this average multiplicity as $d^3n^{B|A}$ and refer to it as the *differential conditional average multiplicity* [5] of B given A. We use the lower-case notation $n^{B|A}$ to explicitly denote a per-event average; while we use the upper-case notation, such as N^A, to denote a full-count multiplicity which is proportional to the size of the full event sample. In the present case $d^3n^{B|A}$ is a per-event average, but only over certain events and not all events; this can be tricky, so we will later make sure that our result is sensible by double-checking three conditions: (i) our result for $d^3n^{B|A}$ should be proportional to the size of the phase-space volume $d^3\mathbf{p}_B$, but (ii) it should be independent of the size of the phase-space volume $d^3\mathbf{p}_A$, and (iii) it should be independent of the overall size of the event sample.

Moving very carefully, now: We denote the number of events in the whole sample as N_{evt}. We denote the total number of A particles created into $d^3\mathbf{p}_A$ around $\mathbf{p}_A$, over the whole event sample, as d^3N^A. Then the probability for such an outcome, over the whole event sample, is d^3N^A/N_{evt}. Similarly, the total number of coincidences – that is, pairs – with A into $d^3\mathbf{p}_A$ around $\mathbf{p}_A$ and B into $d^3\mathbf{p}_B$ around $\mathbf{p}_B$ is denoted d^6N^{AB}; and the probability of such a pair is d^6N^{AB}/N_{evt}. Then we can incarnate Eq. 17 directly and say that the differential conditional average multiplicity should be the ratio of those probabilities:

$$\begin{aligned} d^3n^{B|A} &\equiv \frac{d^6N^{AB}/N_{evt}}{d^3N^A/N_{evt}} \\ &= d^3\mathbf{p}_B \frac{\frac{1}{N_{evt}}\frac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B}}{\frac{1}{N_{evt}}\frac{d^3N^A}{d^3\mathbf{p}_A}} \end{aligned} \tag{18}$$

It should be clear that the form 18 satisfies the the conditions (i)–(iii) listed above, so this is a sensible result for the quantity we seek. If we divide both sides of Eq. 18 by $d^3\mathbf{p}_B$ then we can name the result the *full* conditional average multiplicity

[4]This equation is not a definition of the conditional probability, but can be shown to follow given its above definition. The proof is either trivial or profound, depending on one's outlook.

[5]A conditional multiplicity is also sometimes referred to as a *conditional yield*; the two terms are entirely equivalent. (Also synonymous are the commonly-used term *associated yield* and the somewhat less common *associated multiplicity.*) In this document we also insert "average" rigorously to distinguish from full-count multiplicities; though in common usage "multiplicity" or "yield" is often intended to mean a per-event average implicitly.

$$\frac{d^3 n^{B|A}}{d^3\mathbf{p}_B}(\mathbf{p}_A,\mathbf{p}_B) = \frac{\dfrac{1}{N_{evt}}\dfrac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B}}{\dfrac{1}{N_{evt}}\dfrac{d^3N^A}{d^3\mathbf{p}_A}} \tag{19}$$

where "full" refers to the fact that this is a function defined over the full six phase-space variables $\mathbf{p}_A$ and $\mathbf{p}_B$, as we indicate explicitly.[6] Recalling Eq. 2 we can see immediately that if A and B production are completely un-correlated then the conditional average multiplicity for B given A reduces to just the per-event multiplicity distribution for B alone, which is exactly what one would expect.

So the full conditional average multiplicity as defined in Eq. 19 is a sensible quantity to measure in pairs analysis; it contains similar, though not identical, information to the full correlation function. And just as it was natural to reduce the full correlation function, it is similarly natural to define a *reduced conditional average multiplicity.*

We cannot simply integrate the numerator and denominator of Eq. 19 as we did in Eq. 6, since the denominator of Eq. 19 is a function only of $\mathbf{p}_A$ and so its integral over $\mathbf{p}_B$ could be inifinite. However, with a little care we can write a sensible equivalent result. We break the general full phase-space volume Γ into a range Γ_A over $\mathbf{p}_A$ and a range Γ_B over $\mathbf{p}_B$. If we allow Γ_B to vary as a function of $\mathbf{p}_A$, *ie* $\Gamma_B(\mathbf{p}_A)$, then we can formally cover any arbitrary volume over the full phase space. That is, for any Γ over the full space there is a choice of Γ_A and $\Gamma_B(\mathbf{p}_A)$ such that the two integrations are identical:

$$\int_\Gamma d^3\mathbf{p}_A d^3\mathbf{p}_B = \int_{\Gamma_A} d^3\mathbf{p}_A \int_{\Gamma_B(\mathbf{p}_A)} d^3\mathbf{p}_B \tag{20}$$

With this separation, and also generalizing $\Gamma_A(\alpha)$ and $\Gamma_B(\alpha,\mathbf{p}_A)$ to be functions of some general parameter(s) α, then we can define the reduced conditional average multiplicity as a function of α

$$n^{B|A}_{\mathrm{Reduced}}(\alpha) \equiv \frac{\displaystyle\int_{\Gamma_A(\alpha)} d^3\mathbf{p}_A \int_{\Gamma_B(\alpha,\mathbf{p}_A)} d^3\mathbf{p}_B \,\frac{1}{N_{evt}}\frac{d^6N^{AB}}{d^3\mathbf{p}_A d^3\mathbf{p}_B}}{\displaystyle\int_{\Gamma_A(\alpha)} d^3\mathbf{p}_A \frac{1}{N_{evt}}\frac{d^3N^A}{d^3\mathbf{p}_A}} \tag{21}$$

very much the equivalent of Eq. 6.

In Sections 2.4 and 2.5 we will illustrate a specific example of conditional multiplicity for the case of relative-angle distributions, and compare it to the corresponding example for the correlation function over relative angle.

[6] Note that it makes no difference whether the conditional average multiplicity is defined using per-event averages on the right-hand side of Eq. 19, *ie* with the $1/N_{evt}$ factor, or the full counts over the entire event samples. This is in contrast to the full correlation function in Eq. 3 where only the per-event averages can sensibly be used on the right-hand side.

2 Relative-Aziumthal-Angle Observables

2.1 Angular Correlations

We will now exhibit a specific, concrete implementation of the general form in Eq. 6 to focus on the relative azimuthal angle (ϕ) separation between particles in a pair.

We first choose to specify the particles' momenta $\mathbf{p}_A$ and $\mathbf{p}_B$ in terms of the separate variables (p_A, θ_A, ϕ_A) and (p_B, θ_B, ϕ_B). We then define a *bin* for each A and B as some range over momentum and polar angle; we refer to these ranges generically as $binA$ and $binB$. Then we ask: for pairs with one A particle in $binA$ and one B particle in $binB$, how are they distributed in relative angle $\Delta\phi \equiv (\phi_A - \phi_B)$?

To focus on the azimuthal angle dependence, we will integrate over the (p, θ) bins for the singles and pair distributions:

$$\frac{1}{N_{evt}}\frac{dN^A}{d\phi_A} \equiv \int_{binA} dp_A d\theta_A \frac{1}{N_{evt}}\frac{d^3N^A}{dp_A d\theta_A d\phi_A} \tag{22}$$

$$\frac{1}{N_{evt}}\frac{dN^B}{d\phi_B} \equiv \int_{binB} dp_B d\theta_B \frac{1}{N_{evt}}\frac{d^3N^B}{dp_B d\theta_B d\phi_B} \tag{23}$$

$$\frac{1}{N_{evt}}\frac{d^2N^{AB}}{d\phi_A\, d\phi_B} \equiv \int_{binA} dp_A d\theta_A \int_{binB} dp_B d\theta_B \frac{1}{N_{evt}}\frac{d^6N^{AB}}{dp_A d\theta_A d\phi_A dp_B d\theta_B d\phi_B} \tag{24}$$

These ϕ distributions count only singles and pairs in the specified bins, and so should rightly be noted as specifically defined for $binA$ and $binB$. For convenience, however, we will use the fairly intuitive notation for all ϕ distributions that a label "A" indicates "a particle of type A appearing with (p_A, θ_A) in the range $binA$" — and the same, of course, for B.

We can then define the total count of singles in each bin, and of pairs with one in each bin, over the whole event sample:

$$N^A \equiv \int_0^{2\pi} d\phi_A \frac{dN^A}{d\phi_A} \quad ; \quad N^B \equiv \int_0^{2\pi} d\phi_B \frac{dN^B}{d\phi_B} \tag{25}$$

$$N^{AB} \equiv \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \frac{d^2N^{AB}}{d\phi_A\, d\phi_B} \tag{26}$$

Since the singles distributions must be isotropic with respect to ϕ as measured in the lab (assuming the incoming beams do not have any transverse polarization) the singles ϕ distributions must reduce simply to

$$\frac{1}{N_{evt}}\frac{dN^A}{d\phi_A} = \frac{1}{N_{evt}}\frac{N^A}{2\pi} = \frac{N^A/N_{evt}}{2\pi} \equiv \frac{n^A}{2\pi} \tag{27}$$

$$\frac{1}{N_{evt}}\frac{dN^B}{d\phi_B} = \frac{1}{N_{evt}}\frac{N^B}{2\pi} = \frac{N^B/N_{evt}}{2\pi} \equiv \frac{n^B}{2\pi} \tag{28}$$

where we have now defined $n^A \equiv N^A/N_{evt}$ and $n^B \equiv N^B/N_{evt}$ to be the averge number of A and B particles produced per event into $binA$ and $binB$ (respectively, of course) over the whole event sample. We will also define n^{AB} as the average number of AB pairs per event:

$$n^{AB} \equiv \frac{N^{AB}}{N_{evt}} = \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \, \frac{1}{N_{evt}} \frac{d^2 N^{AB}}{d\phi_A \, d\phi_B} \tag{29}$$

for a full and consistent notation.

The definitions of the ranges $binA$ and $binB$ go most of the way to defining a phase-space volume Γ which would let us define a correlation function. To complete the specification, we will say we are interested in examining pairs with their azimuthal angle difference $\phi_A - \phi_B$ equal to some specific value $\Delta\phi$. The parameter $\Delta\phi$ then takes on the role of α in Eq. 6 and we can define our integration region as

$$\int_{\Gamma(\Delta\phi)} \equiv \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \delta(\Delta\phi - (\phi_A - \phi_B)) \int_{binA} dp_A d\theta_A \int_{binB} dp_B d\theta_B \tag{30}$$

Then, using the notation defined in Eq.'s 22, 23 and 24 the correlation function that results from using form 30 in Eq. 6 will appear, after integrating over the bins, as

$$\begin{aligned}
C(\Delta\phi) \quad &\equiv \quad \frac{\displaystyle\int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \, \delta(\Delta\phi - (\phi_A - \phi_B)) \frac{1}{N_{evt}} \frac{d^2 N^{AB}}{d\phi_A dp_B d\phi_B}}{\displaystyle\int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \delta(\Delta\phi - (\phi_A - \phi_B)) \left(\frac{1}{N_{evt}} \frac{dN^A}{d\phi_A} \frac{1}{N_{evt}} \frac{dN^B}{d\phi_B} \right)} &(31)\\
&= \quad \frac{\displaystyle\int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \, \delta(\Delta\phi - (\phi_A - \phi_B)) \, \frac{1}{N_{evt}} \frac{d^2 N^{AB}}{d\phi_A d\phi_B}}{\displaystyle\frac{n^A}{2\pi} \frac{n^B}{2\pi} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \, \delta(\Delta\phi - (\phi_A - \phi_B))} \\
&= \quad \frac{2\pi}{n^A \, n^B} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \, \delta(\Delta\phi - (\phi_A - \phi_B)) \, \frac{1}{N_{evt}} \frac{d^2 N^{AB}}{d\phi_A d\phi_B} &(32)
\end{aligned}$$

The reduced correlation function $C(\Delta\phi)$ could be called a "relative-azimuthal-angle correlation function" but is more typically called just the *angular correlation function* for short. Though we have not noted it explicitly, it should be clear that each $C(\Delta\phi)$ is defined for one particular choice of particle types A and B along with (p, θ)-ranges $binA$ and $binB$.

The form of Eq. 32 is really quite simple, which makes predicting the angular correlation function within a specific model quite tractable. The main work of the model is in predicting $d^2N^{AB}/d\phi_A d\phi_B$, the joint distribution of ϕ angles for particles in pairs in the bins, after which evaluating the integral in Eq. 32 is typically quite straightforward.

Moral. The moral of the mathematical story thus far is: if you trust your intuition then you may be tempted to jump from a model directly to writing down a predicted correlation function; but you'll always be safe if you start with the pairs' joint ϕ distribution and *then* derive what you expect for a correlation function via Eq. 32 or its equivalent.

2.2 Sum Rule for Angular Correlations

While we observed in Sec. 1.2 that correlation functions in general do not obey any simple sum rules, angular correlation functions do admit at least one useful rule. If we integrate the correlation function $C(\Delta\phi)$ over the full range of $\Delta\phi$ the result has a simple interpretation. Starting from Eq. 32, and with nothing up my sleeve, we have:

$$
\begin{aligned}
\int_0^{2\pi} d(\Delta\phi)\, C(\Delta\phi) &= \int_0^{2\pi} d(\Delta\phi) \frac{2\pi}{n^A\, n^B} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \delta(\Delta\phi - (\phi_A - \phi_B))\, \frac{1}{N_{evt}} \frac{d^2 N^{AB}}{d\phi_A d\phi_B} \\
&= \frac{2\pi}{n^A\, n^B} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \frac{1}{N_{evt}} \frac{d^2 N^{AB}}{d\phi_A d\phi_B} \int_0^{2\pi} d(\Delta\phi)\, \delta(\Delta\phi - (\phi_A - \phi_B)) \\
&= \frac{2\pi}{n^A\, n^B} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \frac{1}{N_{evt}} \frac{d^2 N^{AB}}{d\phi_A d\phi_B} \\
&= 2\pi\, \frac{n^{AB}}{n^A\, n^B} \qquad (33)
\end{aligned}
$$

where we recall from Eq. 29 that n^{AB} is the average number of AB pairs per event, from all combinations of sources. So we can see from Eq. 33 that the integral of $C(\Delta\phi)$ over all $\Delta\phi$ is just equal to 2π times the average number of AB pairs per event, divided by the product of the average number of A and B singles per event. Note that the latter ratio, the rate of pairs divided by the product of rates of singles, is exactly a factorized correlation function as defined in Sec. 1.4, if the two integration regions Γ_A, Γ_B are taken to be the same as $binA$ and $binB$ combined with full range over ϕ_A and ϕ_B.

In principle this rule could provide a very straightforward way to normalize an angular correlation function – see the discussion in Sec. 1.7 – though the effects of "uninteresting" correlations due to effects such as finite centrality spread would, of course, still be present.

2.3 Worked Example: Elliptic Flow

We now illustrate the calculation of a correlation function from a model using the important and time-honored example of elliptic flow.

We can describe the simplest, basic model for elliptic flow in three statements:

(1) On every event there is some azimuthal angle Φ_{RP} which describes the impact parameter direction, also known as the *reaction plane* direction.

(2) Over all events with fixed Φ_{RP} the ϕ distribution of any type or group of particles will be proportional to $dN/d\phi \propto (1 + 2v_2 \cos(2(\phi - \Phi_{RP})))$, for some constant v_2 specific to that type/group. We will refer to v_2 as a *quadrupole coefficient* or *quadrupole strength*, for obvious reasons.

(3) Other than respecting the reaction plane, different groups of particles will have no other source of azimuthal correlation.

We can see that this model fits perfectly the definition of a residual correlation as described in Sec. 1.5, with Φ_{RP} taking on the role of a continuous underlying parameter. Following Eq. 9 and using the notation of Sec. 2.1 we can write the pairs and singles ϕ distributions for fixed Φ_{RP} quite easily

$$\begin{aligned}
\frac{1}{N_{evt}}\frac{dN^A(\Phi_{RP})}{d\phi_A} &= \frac{n^A}{2\pi}\left[1+2v_2^A\cos\left(2(\phi_A-\Phi_{RP})\right)\right] \\
\frac{1}{N_{evt}}\frac{dN^B(\Phi_{RP})}{d\phi_B} &= \frac{n^B}{2\pi}\left[1+2v_2^B\cos\left(2(\phi_B-\Phi_{RP})\right)\right] \\
\frac{1}{N_{evt}}\frac{d^2N^{AB}(\Phi_{RP})}{d\phi_A d\phi_B} &= \frac{n^A\,n^B}{(2\pi)^2}\,\left[1+2v_2^A\cos\left(2(\phi_A-\Phi_{RP})\right)\right]\left[1+2v_2^B\cos\left(2(\phi_B-\Phi_{RP})\right)\right]
\end{aligned} \tag{34}$$

To get the full pairs distribution we just follow Eq. 13: we mutiply Eq. 34 by the distribution of Φ_{RP}, which is clearly $dN/d\Phi_{RP} = 1/2\pi$, and then integrate over Φ_{RP}:

$$\frac{1}{N_{evt}}\frac{d^2N^{AB}}{d\phi_A d\phi_B} = \int d\Phi_{RP}\frac{1}{2\pi}\frac{n^A\,n^B}{(2\pi)^2}\,\left[1+2v_2^A\cos\left(2(\phi_A-\Phi_{RP})\right)\right]\left[1+2v_2^B\cos\left(2(\phi_B-\Phi_{RP})\right)\right] \tag{35}$$

It is left as a (very worthwhile) exercise for the reader to work through the integral in Eq. 35 and verify that the result is:

$$\frac{1}{N_{evt}}\frac{d^2N^{AB}}{d\phi_A d\phi_B} = \frac{n^A\,n^B}{(2\pi)^2}\,\left[1+2v_2^A v_2^B\cos\left(2(\phi_A-\phi_B)\right)\right] \tag{36}$$

We can then write the angular correlation function predicted in the simple elliptic flow model immediately by inserting Eq. 36 into Eq. 32

$$C(\Delta\phi) = \frac{2\pi}{n^A\,n^B}\int_0^{2\pi} d\phi_A\int_0^{2\pi} d\phi_B\,\delta(\Delta\phi-(\phi_A-\phi_B))\,\frac{n^A\,n^B}{(2\pi)^2}\,\left[1+2v_2^A v_2^B\cos\left(2(\phi_A-\phi_B)\right)\right] \tag{37}$$

Another worthwhile exercise is to show that the integration in Eq. 37 results in

$$C(\Delta\phi) = 1+2v_2^A v_2^B\cos\left(2\Delta\phi\right) \tag{38}$$

So we emerge with the non-trivial result that within the simple elliptic flow model the angular correlation function over $\Delta\phi$ will have the same quadrupole shape as the singles distributions did over $\phi-\Phi_{RP}$, but the coefficient of the quadrupole term in the correlation function is basically the product of the two quadrupole coefficients for the singles distributions. Thus, measuring an angular correlation function can access the underlying quadrupole strengths for the elliptic flow of the singles distributions — at least through the product $v_2^A v_2^B$ — without the need to measure a reaction plane direction. Keeping in mind the cautions of Sec. 1.7, we can also see that even if a correlation function of the type in Eq. 38 were measured with only an arbitrary normalization it is still straightforward to extract the quadrupole amplitude, either through a simple fit or as a Fourier moment.

For these reasons, among others, angular correlation analyses are attractive as a means of investigating elliptic flow, and have been put to great use historically in that regard.

2.4 Relative-Angle Conditional Average Multiplicity

We can define the reduced conditional average mutiplicity which corresponds to the angular correlation function $C(\Delta\phi)$, which we will refer to as the *relative-angle conditional average multiplicity* and denote $n^{B|A}(\Delta\phi)$. Following the definition of $\Gamma(\Delta\phi)$ for the angular correlation function in Eq. 30, we can now write the separate integration regions in Eq. 20, including their $\Delta\phi$ dependence:

$$\int_{\Gamma_A(\Delta\phi)} d^3\mathbf{p}_A = \int_0^{2\pi} d\phi_A \int_{binA} dp_A d\theta_A \tag{39}$$

$$\int_{\Gamma_B(\Delta\phi,\mathbf{p}_A)} d^3\mathbf{p}_B = \int_0^{2\pi} d\phi_B \delta(\Delta\phi - (\phi_A - \phi_B)) \int_{binB} dp_B d\theta_B \tag{40}$$

Using the defintions of the bin-integrated ϕ distributions as shown in Eq.'s 22, 23 and 24 we can now write Eq. 21 for the specific case of relative-angle conditional average multiplicty as

$$n^{B|A}(\Delta\phi) \equiv \frac{\int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \delta(\Delta\phi - (\phi_A - \phi_B)) \frac{1}{N_{evt}} \frac{d^2N^{AB}}{d\phi_A d\phi_B}}{\int_0^{2\pi} d\phi_A \frac{1}{N_{evt}} \frac{dN^A}{d\phi_A}} \tag{41}$$

$$= \frac{1}{n^A} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \delta(\Delta\phi - (\phi_A - \phi_B)) \frac{1}{N_{evt}} \frac{d^2N^{AB}}{d\phi_A d\phi_B} \tag{42}$$

Comparing Eq.'s 41 and 42 to Eq.'s 31 and 32 we can see that the relative-angle conditional average multiplicity and the angular correlation function are essentially identical, differing only by a constant factor of $2\pi/n^B$.

From this point on we will concentrate on the angular correlation function, since the relative-angle conditional average multiplicity can always be derived from it if needed via

$$n^{B|A}(\Delta\phi) = \frac{n^B}{2\pi} C(\Delta\phi) \tag{43}$$

2.5 Comparison with other notations

In the case of two-particle correlation analyses the forms in Eq.'s 3 and 6 are the standard and generally agreed-upon definitions of full and reduced correlation functions, so we can be reasonably sure that the results derived here can be compared directly with those of most angular-correlation analyses. To say the same of the conditional multiplicity observable defined here, however, takes a little more care.

The forms in Eq.'s 41 and 42 follow directly from the general definition in Eq. 21, but they may not look immediately familiar even to those who have carried out a conditional multiplicity/yield analysis. One reason for this is that we have proceeded in Sec. 1.8 from a first-principles definition – starting with Eq. 18 – of what could "reasonably" be termed a conditional probability, or multiplicity, while most analyses start from a more operational defintion. It is worth some time to show that the two are, reassuringly, equivalent.

The observable used in most conditional multiplicity analyses is usually denoted

$$\frac{1}{N_{Trigger}} \frac{dN}{d(\Delta\phi)} \tag{44}$$

Here "Trigger" denotes one type or class of particle, and the aim is to quantify the number of particles from a second "Companion" type/class that are observed in coincidence with a "Trigger" particle[7].

Operationally the quantity in (44) is very straightforward to construct. The first factor $1/N_{Trigger}$ is simply the inverse of the total number of "Trigger" particles observed in the entire event sample, irrespective of the presence or absence of "Companion" particles. The second factor, the distribution $dN/d(\Delta\phi)$, counts the number of times dN that a "Companion" particle was observed on the same event as a "Trigger" particle, *and* that the difference between the two particles' aziumthal angles ($\phi_{Companion} - \phi_{Trigger}$) was within the range $d(\Delta\phi)$ around $\Delta\phi$. In effect $dN/d(\Delta\phi)$ describes the distribution of "Trigger-Companion" pairs over the pair variable $\Delta\phi$, and should have the properties of a pairs distribution, *eg* it should integrate to the total number of pairs observed in the event sample.

There is an unfortunate semantic confusion that sometimes arises with the form (44), to wit: some will describe the action of calculating form (44) as "counting the number of Companion particles observed per Trigger particle", which is then elided to "counting the number of Companion particles and then dividing by the number of Trigger particles". The latter description then sometimes inspires the notation, which we would not recommend,

$$\frac{1}{N_{Trigger}} \frac{dN_{Companion}}{d(\Delta\phi)} \quad \text{(Not preferred notation)} \tag{45}$$

The notation used in (45) is misleading, in that it suggests we are counting "Companion" particles – *ie* some kind of singles – while we really should be counting "Trigger-Companion" coincidences, *ie* pairs. The most explicit notation

[7] In other notations the "Trigger" particles are also referred to as "Seed", "Leading" or "Parent" particles; while the "Companion" particles also go by the name "Partner", "Daughter" or "Associated" particles. In this document we will use "Trigger" and "Companion"; but at the time of this writing the naming field is still open, so if you have a better idea then let people know.

$$\frac{1}{N_{Trigger}}\,\frac{dN_{Trigger-Companion\ \ Pair}}{d(\Delta\phi)} \tag{46}$$

would make this unmistakably clear; but since the form (46) might be cumbersome most authors use (44) and leave its equivalence to (46) implicit. Notations such as (45) should be avoided, and the fact should never be lost or concealed that we are counting pairs as the basic objects.

With that said we can now easily make contact with the notation developed in this document, letting our particle selection A play the role of "Trigger" and B play the role of "Companion". Recalling that the per-event and total singles multiplicites are related by $N^A = n^A \cdot N_{evt}$ we can re-write our definition for conditional multiplicity in Eq. 42 suggestively as

$$n^{B|A}(\Delta\phi) = \frac{1}{N^A}\int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\,\delta(\Delta\phi - (\phi_A - \phi_B))\,\frac{d^2N^{AB}}{d\phi_A d\phi_B} \tag{47}$$

After a moment's reflection it should be clear that the double-integral factor on the RHS in Eq. 47 is exactly equal to the pairs relative-angle distribution $dN^{AB}/d(\Delta\phi)$, with our usual definition $\Delta\phi = \phi_A - \phi_B$.

We can now state the equivalence between the notations explicitly:

$$n^{B|A}(\Delta\phi) = \frac{1}{N^A}\,\frac{dN^{AB}}{d(\Delta\phi)} \tag{48}$$

For the convenience of the reader who may prefer one or the other of these notations we will carry both to represent the relative-angle conditional average multiplicity for the rest of this document. For example, we will re-state the relationship to the reduced angular correlation function in Eq. 43 now with the more common notation

$$\frac{1}{N^A}\,\frac{dN^{AB}}{d(\Delta\phi)} = \frac{n^B}{2\pi}\,C(\Delta\phi) \tag{49}$$

just so you don't forget it (!).

Lastly, we can make the interesting observation that in a conditional multiplicity analysis it is really not that important which particle we call "Trigger" and which "Companion". More exactly, a measurement of the multiplicity of B conditioned on the presence of A contains essentially the same information as a measurement of the reverse, the multiplicity of A conditioned on the presence of B. Re-writing Eq.'s 48 and 49 for the reverse case we have

$$n^{A|B}(\Delta\phi) = \frac{1}{N^B}\,\frac{dN^{AB}}{d(\Delta\phi)} = \frac{n^A}{2\pi}\,C(\Delta\phi) \tag{50}$$

Combining Eq.'s 49 and 50 it then becomes clear that the two relative-angle conditional average multiplicities are very closely related:

$$n^{B|A}(\Delta\phi) = \frac{1}{N^A}\frac{dN^{AB}}{d(\Delta\phi)} = \left(\frac{n^B}{n^A}\right)\,\frac{1}{N^B}\frac{dN^{AB}}{d(\Delta\phi)} = \left(\frac{n^B}{n^A}\right)\,n^{A|B}(\Delta\phi) \tag{51}$$

So as long as the average (or total) bin multiplicities for A and B are known the two measurements yield entirely equivalent information, and which one we choose to compile is essentially a matter of taste.

2.6 Sum Rule for Relative-Angle Conditional Multiplicity

In Sec. 2.2 we derived a simple sum rule for the angular correlation function, and since the two are so closely related we should expect there to be a corresponding rule for the relative-angle conditional averge multiplicity. Folding Eq.'s 48 and 49 into Eq. 33 we have

$$\int_0^{2\pi} d(\Delta\phi)\,n^{B|A}(\Delta\phi) = \int_0^{2\pi} d(\Delta\phi)\,\frac{1}{N^A}\,\frac{dN^{AB}}{d(\Delta\phi)} = \frac{n^{AB}}{n^A} \tag{52}$$

This is an eminently sensible result. Once we integrate over all $\Delta\phi$ we have counted all the "Trigger"–"Companion" pairs, and so the final result should be the ratio of pairs per "Trigger". In practice this rule may or may not prove useful, since it may be satisfied by construction depending on how the conditional multiplicity is compiled. In this document we will use the rule to double-check our final results for the two-source model in Sec. 3.11.

3 The Two-Source Model: Flow and Jets

3.1 Singles Distribution for Flow Source

We define N^A_{Flow} as the total number of A particles produced into $binA$ from the flow source over the whole event sample, and $n^A_{\text{Flow}} \equiv N^A_{\text{Flow}}/N_{evt}$ as the average number number per event; and, of course, we also define N^B_{Flow} and n^B_{Flow} equivalently for B particles from the flow source. With these, the rest is essentially identical to what was described in Sec. 2.3: the singles distribution is controlled by the underlying parameter Φ_{RP}, the reaction plane direction

$$\begin{aligned}
\frac{1}{N_{evt}}\frac{dN^A_{\text{Flow}}(\Phi_{RP})}{d\phi_A} &= \frac{n^A_{\text{Flow}}}{2\pi}\left[1 + 2v_2^{FlowA}\cos\left(2(\phi_A - \Phi_{RP})\right)\right] \\
\frac{1}{N_{evt}}\frac{dN^B_{\text{Flow}}(\Phi_{RP})}{d\phi_B} &= \frac{n^B_{\text{Flow}}}{2\pi}\left[1 + 2v_2^{FlowB}\cos\left(2(\phi_B - \Phi_{RP})\right)\right]
\end{aligned} \tag{53}$$

We can make a quick check on the normalization (*ie* making sure we have all our 2π's in place) by calculating the full singles distribution. As per Eq. 12 we multiply (53) by the distribution $dN/d\Phi_{RP} = 1/2\pi$ and integrate over Φ_{RP}. The result is $(1/N_{evt})dN^A_{\text{Flow}}/d\phi_A = n^A_{\text{Flow}}/2\pi$, which is exactly what it should be, isotropic and with a total integral of just n^A_{Flow}.

3.2 Singles Distribution for Jet Source

The singles distribution for particles from a jet source is slightly more complicated, since they are controlled by two underlying parameters: Φ_{Jet} the jet axis direction, and Φ_{RP} the reaction plane direction. Our model for jet fragmentation, and for jet propogation potentially respecting the reaction plane, is embodied in two statements:

(1) For jets emerging at an azimuthal direction Φ_{Jet} the fragments which are type A particles produced into $binA$ are distributed in aziumth according to some function $J^A(\phi_A - \Phi_{Jet})$. We assume this function $J^A()$ to be symmetric, single-peaked at 0, and to have a value of zero for $|\phi_A - \Phi_{Jet}| > \pi/2$, *ie* there are no jet fragments produced at angles more than 90^o away from the jet direction. And, of course, there is an equivalent function $J^B(\phi_B - \Phi_{Jet})$ describing the aziumthal distribution of B fragments. Further, we also stipulate for convenience that all $J()$ functions are normalized to a total integral of 1.

(2) There are many reasonable possible pictures for how jet production could “feel” the collision geometry through interaction with some non-aziumthally-symmetric created medium. Here we embody those possible effects as one simple assumption: for jets emerging at direction Φ_{Jet}, the number of A fragments observed in $binA$ will be modulated by a function $(1 + 2\langle v_2^{JetA}\rangle \cos 2(\Phi_{Jet} - \Phi_{RP}))$ for some constant $\langle v_2^{JetA}\rangle$ which is specific to the definition of $binA$. The behavior of B fragments, of course, is the same but controlled by a corresponding constant $\langle v_2^{JetB}\rangle$.

We use the bracket $\langle\rangle$ notation to emphasize that the effective quadrupole strength $\langle v_2^{JetA}\rangle$ is not the strength for any particular jet, but is an *average* over all jets that produce A fragments into $binA$, where the average must extend over all jet energies, polar angles, parton types, etc.

We now define, exactly as in Sec. 3.1, $N^A_{\rm Jet}$ and $N^B_{\rm Jet}$ to be the total number of A particles in $binA$, and B particles in $binB$, which are produced by jet sources; and we similarly define $n^A_{\rm Jet}$ and $n^B_{\rm Jet}$ to be their per-event averages, respectively. With these defined we can now write the singles distributions for give values of Φ_{Jet} and Φ_{RP} as

$$\begin{aligned}
\frac{1}{N_{evt}}\frac{dN^A_{\rm Jet}(\Phi_{Jet},\Phi_{RP})}{d\phi_A} &= n^A_{\rm Jet}\, J^A(\phi_A - \Phi_{Jet})\,\left[1 + 2\langle v_2^{JetA}\rangle \cos\left(2(\Phi_{Jet} - \Phi_{RP})\right)\right] \\
\frac{1}{N_{evt}}\frac{dN^B_{\rm Jet}(\Phi_{Jet},\Phi_{RP})}{d\phi_B} &= n^B_{\rm Jet}\, J^B(\phi_B - \Phi_{Jet})\,\left[1 + 2\langle v_2^{JetB}\rangle \cos\left(2(\Phi_{Jet} - \Phi_{RP})\right)\right] \qquad (54)
\end{aligned}$$

It is useful to derive the distribution with only Φ_{RP} as the controlling parameter, by taking the singles distribution multiplying by $dn/d\Phi_{Jet} = 1/2\pi$ and integrating over Φ_{Jet}. We will work through the result for A particles, leaving the parallel conclusion for B implicit.

$$\frac{1}{N_{evt}}\frac{dN^A_{\rm Jet}(\Phi_{RP})}{d\phi_A} = \int d\Phi_{Jet}\,\frac{1}{2\pi}\,\frac{1}{N_{evt}}\frac{dN^A(\Phi_{Jet},\Phi_{RP})}{d\phi_A}$$

$$
\begin{aligned}
&= \frac{n^A_{\text{Jet}}}{2\pi} \int d\Phi_{Jet} J^A(\phi_A - \Phi_{Jet}) \left[1 + 2\langle v_2^{JetA}\rangle \cos\left(2(\Phi_{Jet} - \Phi_{RP})\right)\right] \\
&= \frac{n^A_{\text{Jet}}}{2\pi} \left[\int d\Phi_{Jet} J^A(\phi_A - \Phi_{Jet}) + \right. \\
&\qquad \left. 2\langle v_2^{JetA}\rangle \int d\Phi_{Jet} J^A(\phi_A - \Phi_{Jet}) \cos\left(2(\Phi_{Jet} - \Phi_{RP})\right)\right]
\end{aligned} \tag{55}
$$

The first integral in Eq. 55 evaluates to just 1 by the definition of $J()$. The second integral is a little trickier, but by expanding the cosine and making the change of variable $x \equiv (\phi_A - \Phi_{RP})$ we arrive at

$$
\frac{1}{N_{evt}} \frac{dN^A_{\text{Jet}}(\Phi_{RP})}{d\phi_A} = \frac{n^A_{\text{Jet}}}{2\pi} \left[1 + 2\langle v_2^{JetA}\rangle j_2^A \cos\left(2(\phi_A - \Phi_{RP})\right)\right] \tag{56}
$$

where the constant j_2^A is just the second cosine Fourier moment of $J^A()$

$$
j_2^A \equiv \int J^A(x)\, \cos(2x)\, dx \tag{57}
$$

If the function $J^A()$ is narrowly peaked, then the value of j_2^A will be very close to 1.

We can see that the singles distribution of jet fragments for fixed Φ_{RP} follows the elliptic-flow-like quadrupole pattern perfectly, with the quadrupole strength being equal to the average strength for jets which generate A particles in $binA$, modified only slightly by the factor j_2^A which will typically be close to 1. So in this model the quadrupole strength of jet fragments quite directly reflects the quadrupole pattern of the jets themseleves.

3.3 Combined Singles Distribution

It is useful to write the full singles distribution, for the sum of both sources, for given Φ_{RP}. First we note that $N^A = N^A_{\text{Flow}} + N^A_{\text{Jet}}$ by definition, and similarly $n^A = n^A_{\text{Flow}} + n^A_{\text{Jet}}$ by defintion (with the same, of course, true for B). Then we can write the sum of Eq.'s 53 and 56 compactly as

$$
\frac{1}{N_{evt}} \frac{dN^A_{\text{Jet}}(\Phi_{RP})}{d\phi_A} + \frac{1}{N_{evt}} \frac{dN^A_{\text{Flow}}(\Phi_{RP})}{d\phi_A} = \frac{n^A}{2\pi} \left[1 + 2V_2^A \cos(2(\phi_A - \Phi_{RP}))\right] \tag{58}
$$

where V_2^A is the constant

$$
V_2^A \equiv \frac{n^A_{\text{Flow}}}{n^A} v_2^{FlowA} + \frac{n^A_{\text{Jet}}}{n^A} j_2^A \langle v_2^{JetA}\rangle \tag{59}
$$

We can see from Eq. 58 that the full sum singles distribution for fixed Φ_{RP} follows a quadrupole shape, and from Eq. 59 that its quadrupole strength V_2^A is just the weighted average of the quadrupole strengths of the flow source and jet source separately. This is a standard result when adding two distributions each with some quadrupole strength, so we should not be surprised — instead, relieved — to recover it here.

Of course (as almost goes without saying), the same thing will be true for the full sum singles distribution of B particles, with the corresponding net quadrupole strength V_2^B.

3.4 Sum Over Pair Types

In the angular correlation function in Equation 32 the pairs distribution $d^2N^{AB}/d\phi_A d\phi_B$, of course, counts all pairs from all combinations of sources. If we can partition the pairs distribution into a sum over pairs of different, disjoint types then we can also write the correlation function as a sum of terms

$$\frac{d^2N^{AB}}{d\phi_A d\phi_B} = \sum_{\text{PairType}} \frac{d^2N^{AB}_{\text{PairType}}}{d\phi_A d\phi_B} \tag{60}$$

$$C(\Delta\phi) = \sum_{\text{PairType}} \frac{2\pi}{n^A\, n^B} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \delta(\Delta\phi - (\phi_A - \phi_B))\, \frac{1}{N_{evt}} \frac{d^2N^{AB}_{\text{PairType}}}{d\phi_A d\phi_B} \tag{61}$$

We will calculate a term in this sum for each of the possible pair types in the two-source model in the following Sections, and then exhibit the overall sum of Eq. 61 in Section 3.10.

3.5 Pairs Type I: Flow-Flow

The simplest type of pair is when both A and B particles are from the flow source. The singles distributions are controlled by Φ_{RP} and the joint distribution is

$$\frac{1}{N_{evt}} \frac{d^2N^{AB}_{\text{Flow}-\text{Flow}}}{d\phi_A d\phi_B} = \int d\Phi_{RP} \frac{1}{2\pi} \frac{1}{N_{evt}} \frac{dN^A_{\text{Flow}}(\Phi_{RP})}{d\phi_A} \frac{1}{N_{evt}} \frac{dN^B_{\text{Flow}}(\Phi_{RP})}{d\phi_B}$$

The forms for the singles distributions from the flow source are shown in Eq. 53, and since they both follow the form of a quadrupole distribution, we know that this is the same as the problem we solved before in Section 2.3. The calculation for the term in Eq. 61 follows exactly the same path, and so we will only quote the result here; the term in Eq. 61 for the Flow-Flow pair type is

$$\begin{aligned} & \frac{2\pi}{n^A\, n^B} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \delta(\Delta\phi - (\phi_A - \phi_B))\, \frac{1}{N_{evt}} \frac{d^2N^{AB}_{\text{Flow}-\text{Flow}}}{d\phi_A d\phi_B} \\ = {} & \frac{n^A_{\text{Flow}}\, n^B_{\text{Flow}}}{n^A\, n^B} \left[1 + 2v_2^{FlowA} v_2^{FlowB} \cos(2\Delta\phi)\right] \end{aligned} \tag{62}$$

3.6 Pairs Type II: Flow-Jet

Here we take first the case where particle A is from the flow source and B is from a jet source, and work it out in detail. When we write the final form for the term that enters Eq. 61 we will add the reverse case as well.

In principle the Flow-Jet pair shows only residual correlation, with the controlling parameters being Φ_{RP} and Φ_{Jet}. So we can calculate the joint distribution in the usual way, by taking the singles distributions from Eq. 53 for A and from Eq. 54 for B, multiplying by $dn/d\Phi_{RP}$ and $dn/d\Phi_{Jet}$, and then integrating over Φ_{RP} and Φ_{Jet}

$$\frac{1}{N_{evt}}\frac{d^2N^{AB}_{\text{FlowA}-\text{JetB}}}{d\phi_A d\phi_B} = \iint d\Phi_{RP}\, d\Phi_{Jet}\, \frac{1}{(2\pi)^2}\, \frac{1}{N_{evt}}\frac{dN^A_{\text{Flow}}(\Phi_{RP})}{d\phi_A}\, \frac{1}{N_{evt}}\frac{dN^B_{\text{Jet}}(\Phi_{Jet}, \Phi_{RP})}{d\phi_B} \tag{63}$$

The integration over Φ_{Jet} affects only the Jet-B factors, and we know the result of this integration from Eq. 56 in Section 3.2. So the joint distribution now reduces to

$$\frac{1}{N_{evt}}\frac{d^2N^{AB}_{\text{FlowA}-\text{JetB}}}{d\phi_A d\phi_B} = \int d\Phi_{RP}\, \frac{1}{2\pi}\, \frac{1}{N_{evt}}\frac{dN^A_{\text{Flow}}(\Phi_{RP})}{d\phi_A}\, \frac{1}{N_{evt}}\frac{dN^B_{\text{Jet}}(\Phi_{RP})}{d\phi_B} \tag{64}$$

The two forms for these singles distributions are in Eq. 53 and Eq. 56 and we can see that they each have exactly the form of a quadrupole modulation. So Eq. 64 has exactly the same form as Eq. 35, just as the Flow-Flow pairs did, and so we can go straight to the corresponding result for the term in the correlation function. Adding both the Jet-A—Flow-B term and its reverse, the full term in Eq. 61 for the Jet-Flow pair type is

$$\begin{aligned}
&\frac{2\pi}{n^A\, n^B}\int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \delta(\Delta\phi - (\phi_A - \phi_B))\, \frac{1}{N_{evt}}\frac{d^2N^{AB}_{\text{Jet}-\text{Flow}}}{d\phi_A d\phi_B} \\
&= \frac{n^A_{\text{Flow}}\, n^B_{\text{Jet}}}{n^A\, n^B}\left[1 + 2v_2^{FlowA} j_2^B \langle v_2^{JetB}\rangle \cos(2\Delta\phi)\right] + \\
&\quad \frac{n^A_{\text{Jet}}\, n^B_{\text{Flow}}}{n^A\, n^B}\left[1 + 2j_2^A \langle v_2^{JetA}\rangle v_2^{FlowB} \cos(2\Delta\phi)\right]
\end{aligned} \tag{65}$$

3.7 Pairs Type III: Unrelated Jets

Pairs of particles when each is from separate, un-related jets have a very similar form to the Flow-Flow and Jet-Flow cases. In principle the pair distribution is controlled by three underlying parameters, the two jet axes Φ_{JetA} and Φ_{JetB} and the reaction plane direction Φ_{RP}, and so the distribution is calculated as

$$\begin{aligned}
&\frac{1}{N_{evt}}\frac{d^2N^{AB}_{\text{UnrelatedJets}}}{d\phi_A d\phi_B} \\
&= \iiint d\Phi_{RP}\, d\Phi_{JetA}\, d\Phi_{JetB}\, \frac{1}{(2\pi)^3}\, \frac{1}{N_{evt}}\frac{dN^A_{\text{Jet}}(\Phi_{JetA}, \Phi_{RP})}{d\phi_A}\, \frac{1}{N_{evt}}\frac{dN^B_{\text{Jet}}(\Phi_{JetB}, \Phi_{RP})}{d\phi_B}
\end{aligned} \tag{66}$$

The singles distributions are shown in Eq. 54; but as before, the integrations over Φ_{JetA} and Φ_{JetB} reduce the calculation of the pairs distribution down to

$$\frac{1}{N_{evt}}\frac{d^2N^{AB}_{\text{UnrelatedJets}}}{d\phi_A d\phi_B} = \int d\Phi_{RP}\frac{1}{2\pi}\,\frac{1}{N_{evt}}\frac{dN^A_{\text{Jet}}(\Phi_{RP})}{d\phi_A}\frac{1}{N_{evt}}\frac{dN^B_{\text{Jet}}(\Phi_{RP})}{d\phi_B} \tag{67}$$

The singles distributions after integrating over the Φ_{Jet}'s are shown in Eq. 56; and, as in the preceding two sections, they are both of the quadrupole type which lets us move directly to the answer. The term in Eq. 61 for the Unrelated Jet pair type is

$$\begin{aligned}
&\frac{2\pi}{n^A\, n^B}\int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \delta(\Delta\phi - (\phi_A - \phi_B))\,\frac{1}{N_{evt}}\frac{d^2N^{AB}_{\text{UnrelatedJets}}}{d\phi_A d\phi_B} \\
&\quad = \frac{n^A_{\text{Jet}}\, n^B_{\text{Jet}}}{n^A\, n^B}\left[1 + 2\, j_2^A\langle v_2^{JetA}\rangle\, j_2^B\langle v_2^{JetB}\rangle\cos(2\Delta\phi)\right]
\end{aligned} \tag{68}$$

3.8 Pairs Type IV: Jet Fragments, Same Side

The first three pair types discussed so far could all be described as experiencing only residual correlations between A and B particles, and their terms in the correlation function all had essentially the same form. The next pair type, where both particles are fragments from the same jet, will be a little more complicated.

Production of A and B particles as fragments from the same jet cannot be said to show purely residual correlation, since even for a jet of fixed energy and direction the production of A and B particles into $binA$ and $binB$ are not uncorrelated. The production of an A into $binA$ from a jet may make the production of a B into $binB$ from the same jet more likely, or less likely, or neither! The exact answer depends very sensitively on the bin definitions and the physics of jet fragmentation, which is not well-understood and is not even that well-studied for the production of multiple fragments.

We will encapsulate our ignorance into a new variable: we define $N^{AB}_{\text{JetSameSide}}$ as the total number of A,B pairs produced into $binA$ and $binB$ by fragmentation from the same jet, over the whole event sample. And, naturally, we define its per-event average as $n^{AB}_{\text{JetSameSide}} \equiv N^{AB}_{\text{JetSameSide}}/N_{evt}$. So whatever form we derive for the joint ϕ distribution $(1/N_{evt})d^2N^{AB}_{\text{JetSameSide}}/d\phi_A d\phi_B$ we can set its normalization by requiring its integral over ϕ_A and ϕ_B to be equal to $n^{AB}_{\text{JetSameSide}}$.

We now make the simplifying assumption, commonly used, that the ϕ_A distribution of particle A is un-affected by the presence or absence of a B fragment from the same jet, and vice versa. This amounts to two statements: (1) The particles A and B show only a residual correlation in ϕ, as both respect the parent jet axis but are otherwise uncorrelated; and (2) The ϕ distribution of a particle with respect to the jet axis in pair events has the same shape as its singles distribution. We cannot justify this assumption from any first principles — fragmentation is not that well understood — but it is not out of line with what models for fragmentation do exist, and is probably quite reasonable at the level of our farily simple two-source picture here.

We will also allow for the possiblity, as with the jet singles distributions, that the jet producing the pair had some interaction with the collision medium that allows it to "feel" the event plane. As before, we will model this effect by saying that the number of produced pairs will be modulated by a quadrupole shape $\left[1 + 2\langle v_2^{JetAB}\rangle \cos(2(\Phi_{Jet} - \Phi_{RP}))\right]$. The quadrupole coefficient is an now average over jets which produce A-B fragment pairs, similar to the coefficients that showed up in singles production; but there is *no* simple relationship between the quadrupole strengths for singles from jets and for pairs from jets (fortunately, this will not turn out to matter).

At least, we can now solve the problem! With the definition and assumption above, we can say that the pairs distribution for fixed jet axis direction Φ_{Jet} and reaction plane direction Φ_{RP} should follow

$$\begin{aligned} &\frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\text{JetSameSide}}(\Phi_{Jet}, \Phi_{RP})}{d\phi_A d\phi_B} \\ &\quad = n^{AB}_{\text{JetSameSide}}\, J^A(\phi_A - \Phi_{Jet})\, J^B(\phi_B - \Phi_{Jet})\, \left[1 + 2\langle v_2^{JetAB}\rangle \cos(2(\Phi_{Jet} - \Phi_{RP}))\right] \end{aligned} \tag{69}$$

As usual, we get the full joint distribution by integrating over Φ_{Jet} and Φ_{RP} including their distributions, each $1/2\pi$:

$$\frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\text{JetSameSide}}}{d\phi_A d\phi_B} = \iint d\Phi_{RP}\, d\Phi_{Jet}\, \frac{1}{(2\pi)^2} \frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\text{JetSameSide}}(\Phi_{Jet}, \Phi_{RP})}{d\phi_A d\phi_B} \tag{70}$$

We can see from the form in Eq. 69 that Φ_{RP} appears only in the quadrupole-shape factor, and so it makes sense to carry out the Φ_{RP} integral first. The integral of the cosine term will vanish, and the integral over 1 in that factor will just produce a factor of 2π. With only the integral over Φ_{Jet} remaining we have

$$\frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\text{JetSameSide}}}{d\phi_A d\phi_B} = \frac{n^{AB}_{\text{JetSameSide}}}{2\pi} \int d\Phi_{Jet}\, J^A(\phi_A - \Phi_{Jet})\, J^B(\phi_B - \Phi_{Jet}) \tag{71}$$

At this point we can make the change of variables $x \equiv \Phi_{Jet} - \phi_B$ and taking advantage of the fact that the $J()$ functions are symmetric around 0, re-write Eq. 71 as

$$\frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\text{JetSameSide}}}{d\phi_A d\phi_B} = \frac{n^{AB}_{\text{JetSameSide}}}{2\pi} \int dx\, J^A((\phi_A - \phi_B) - x)\, J^B(x) \tag{72}$$

We can recognize the integral over x as simply the convolution of J^A with J^B, evaluated at $(\phi_A - \phi_B)$. Using the standard notation $J^A \circ J^B$ for the convolution of the two functions, we finally arrive at the joint distribution:

$$\frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\text{JetSameSide}}}{d\phi_A d\phi_B} = \frac{n^{AB}_{\text{JetSameSide}}}{2\pi} J^A \circ J^B(\phi_A - \phi_B) \tag{73}$$

We mentioned earlier in this Section that our definition of $n^{AB}_{\text{JetSameSide}}$ requries that the joint ϕ distribution must have a total integral equal to exactly $n^{AB}_{\text{JetSameSide}}$. We can double-check our form defined in Eq. 69 by noting that our joint distribution in Eq. 73 clearly satisfies this requirement[8].

[8] It is useful to recall here that the integral of a convolution is just the product of the integrals of the original functions. Since the $J^A()$ and $J^B()$ functions are both normalized to 1 by definition, we know that their convolution is also normalized to 1.

With the form in Eq. 73 for the joint distribution, deriving the term in the sum of Eq. 61 for the Jet-Same-Side pair type is straightforward:

$$\begin{aligned}
&\frac{2\pi}{n^A\, n^B}\int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B\, \delta(\Delta\phi - (\phi_A - \phi_B))\, \frac{1}{N_{evt}}\frac{d^2 N^{AB}_{\text{JetSameSide}}}{d\phi_A d\phi_B} \\
&= \frac{2\pi\, n^{AB}_{\text{JetSameSide}}}{n^A\, n^B} J^A \circ J^B(\Delta\phi) \qquad (74)
\end{aligned}$$

3.9 Pairs Type V: Dijet Fragments, Opposite Side

The most complicated case we will consider in the two-source model is when the A and B particles are fragments from different jets, but the jets are a dijet pair. This means that the two jet axes Φ_{JetA} and Phi_{JetB} will be very close to opposite, or back-to-back, though not perfectly so. We will define the degree to which the two jet axes are not perfectly opposed as ψ^{AB}

$$\psi^{AB} \equiv \Phi_{JetA} - \Phi_{JetB} - \pi \qquad (75)$$

This quantity is often called the "acoplanarity" between the two jets. In this case it describes specifically those di-jet pairs which create A particles into $binA$ and B particles in to $binB$. We can encapsulate our ignorance of di-jet production simply by saying that ψ^{AB} is distributed according to some shape

$$\frac{dn}{d\psi^{AB}} \equiv D^{AB}(\psi^{AB}) \qquad (76)$$

We will assume that the function $D^{AB}()$, like the functions $J^A()$ and $J^B()$, is single-peaked and symmetric around 0, and further stipulate that it is normalized to an integral of 1. Clearly the shape of $D^{AB}()$ is very specific to the definitions of $binA$ and $binB$, and is worthy as an object of study in itself.

We also don't know a priori how many pairs will come from di-jets, and there is no simple relation between the number of jet singles and the number of dijet pairs without more knowledge of jet production. So, as in the previous Section, we will encapsulate this ignorance by defining $N^{AB}_{\text{Di-Jet}}$ as the total number of A,B pairs produced into $binA$ and $binB$ by fragmentation from jets in a di-jet, over the whole event sample, and $n^{AB}_{\text{Di-Jet}} \equiv N^{AB}_{\text{Di-Jet}}/N_{evt}$. as its per-event average.

If we assume that the two jets fragment independently once their axes are known, then we can go ahead and write the joint ϕ distribution. The controlling parameters[9] are now Φ_{JetA}, ψ^{AB} and of course Φ_{RP}; for given values of the parameters the joint ϕ distribution is

$$\frac{1}{N_{evt}}\frac{d^2 N^{AB}_{\text{Di-Jet}}(\Phi_{JetA}, \psi^{AB}, \Phi_{RP})}{d\phi_A d\phi_B} =$$

[9] Note that this is one particular choice for the controlling parameters, and others are possible. Certainly Φ_{JetB}, ψ^{AB} and Φ_{RP} would be equivalent. The set Φ_{JetA}, Φ_{JetB}, and Φ_{RP} is also possible in principle, but we instead choose ψ^{AB} as an independent parameter because we know its distribution explicitly.

$$\eta^{AB} n^{AB}_{\mathrm{Di-Jet}}\, J^A(\phi_A - \Phi_{JetA})\, J^B(\phi_B - (\Phi_{JetA} - \psi^{AB} - \pi)) \;\times$$
$$\Big[1 + 2\langle v_2^{DiJetA}\rangle \cos(2(\Phi_{JetA} - \Phi_{RP}))\Big] \Big[1 + 2\langle v_2^{DiJetB}\rangle \cos(2((\Phi_{JetA} - \psi^{AB} - \pi) - \Phi_{RP}))\Big] \quad (77)$$

where we have made use of the substitution $\Phi_{JetB} = \Phi_{JetA} - \psi^{AB} - \pi$ implicit in Eq. 75.

Note that we have also introduced the normalization constant η^{AB} at the front of Eq. 77. Later we will set the value of η^{AB} to ensure that the joint per-event distribution integrates to exactly $n^{AB}_{\mathrm{Di-Jet}}$. Why do we need to do this here, when we didn't need it for any of the earlier terms? The short answer is that the normalization for all the earlier terms was manifest. The terms for pair types I, II and III were all strictly residual correlations and their normalizations followed from the singles normalizations; the term for pair type IV as shown in Eq.69 integrates to the number of pairs correctly without any additional factors. Only the di-jet case has a non-trivial normalization, but don't worry: the normalization here is *almost* trivial. The constant η^{AB} will turn out to be almost exactly equal to 1, and later we will absorb it into another defined quantity anyway.

We then get the full joint distribution, as usual, by integrating over the parameters after multiplying by their individual normalized distributions

$$\frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\mathrm{Di-Jet}}}{d\phi_A d\phi_B} = \iiint d\Phi_{RP}\, d\Phi_{JetA}\, d\psi^{AB}\, \frac{1}{(2\pi)^2} D^{AB}(\psi^{AB})\, \frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\mathrm{Di-Jet}}(\Phi_{Jet}, \psi^{AB}, \Phi_{RP})}{d\phi_A d\phi_B} \quad (78)$$

The integral resulting from inserting Eq 77 into Eq. 78 may look intimidating, but with a little care it can be handled straightforwardly.

We have included in Eq. 77 the factors for quadrupole modulation of the jets in the di-jet. We have denoted the quadrupole strengths for the jets in the di-jet as $\langle v_2^{DiJetA}\rangle$ and $\langle v_2^{DiJetB}\rangle$; note that these are *not* the same as the quadrupole strengths of the single jets $\langle v_2^{JetA}\rangle$ and $\langle v_2^{JetB}\rangle$ that we defined earlier, since the sample of jets in di-jets producing A–B pairs is *not* the same as the sample of jets which produce A and B particles singly.

While this is formally correct within our model, it turns out almost not to matter at all! We can see this by first carrying out the integral over Φ_{RP} which involves only the factors on the last line of Eq. 77

$$\int d\Phi_{RP} \Big[1 + 2\langle v_2^{DiJetA}\rangle \cos(2(\Phi_{JetA} - \Phi_{RP}))\Big] \Big[1 + 2\langle v_2^{DiJetB}\rangle \cos(2((\Phi_{JetA} - \psi^{AB} - \pi) - \Phi_{RP}))\Big] = 2\pi \Big[1 + 2\langle v_2^{DiJetA}\rangle \langle v_2^{DiJetB}\rangle \cos(2\psi^{AB})\Big] \quad (79)$$

We observe two features of this factor: (1) The value of a product of two v_2's is typically a small number, so we expect that the cosine term will be a small correction that we might even be justified in ignoring;

and (2) Since we know that ψ^{AB} will typically be very close to 0, depending on the width of the D^{AB} distribution, the cosine function will always be very close to 1, and so the whole factor will be very close to constant. We might then be justified in taking it out of the later integrals altogether as just 2π plus a small correction, though for completeness we will keep the full expression here and forge ahead.

After the integration over Φ_{RP} the remaining expression becomes

$$\begin{aligned} & \frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\mathrm{Di-Jet}}}{d\phi_A d\phi_B} \\ = & \frac{\eta^{AB} n^{AB}_{\mathrm{Di-Jet}}}{2\pi} \iint d\Phi_{JetA}\, d\psi^{AB}\, D^{AB}(\psi^{AB})\, J^A(\phi_A - \Phi_{JetA})\, J^B(\phi_B - (\Phi_{JetA} - \psi^{AB} - \pi)) \\ & \times \left[1 + 2\langle v_2^{DiJetA}\rangle\langle v_2^{DiJetB}\rangle \cos(2\psi^{AB})\right] \end{aligned} \tag{80}$$

Now we can carry out the integral over Φ_{JetA}, which involves only the $J()$ functions. Using the change of variables $x \equiv \phi_A - \Phi_{JetA}$ we have

$$\begin{aligned} & \int d\Phi_{JetA} J^A(\phi_A - \Phi_{JetA})\, J^B(\phi_B - (\Phi_{JetA} - \psi^{AB} - \pi)) \\ & = \int dx J^A(x)\, J^B((\phi_A - \phi_B - \psi^{AB} - \pi) - x) \\ & = J^A \circ J^B(\phi_A - \phi_B - \psi^{AB} - \pi) \end{aligned} \tag{81}$$

This is just the convolution of J^A and J^B again, now evaluated at $\phi_A - \phi_B - \psi^{AB} - \pi$. With Eq. 81 in hand we have only the integration over ψ^{AB} remaining:

$$\begin{aligned} \frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\mathrm{Di-Jet}}}{d\phi_A d\phi_B} &= \frac{\eta^{AB} n^{AB}_{\mathrm{Di-Jet}}}{2\pi} \int d\psi^{AB}\, D^{AB}(\psi^{AB})\, J^A \circ J^B(\phi_A - \phi_B - \psi^{AB} - \pi) \\ & \qquad \times \left[1 + 2\langle v_2^{DiJetA}\rangle\langle v_2^{DiJetB}\rangle \cos(2\psi^{AB})\right] \end{aligned} \tag{82}$$

$$= \frac{n^{AB}_{\mathrm{Di-Jet}}}{2\pi} \int d\psi^{AB}\, E^{AB}(\psi^{AB})\, J^A \circ J^B(\phi_A - \phi_B - \psi^{AB} - \pi) \tag{83}$$

where we have now defined the function $E^{AB}(\psi^{AB})$ to be a very small variation on $D^{AB}(\psi^{AB})$

$$E^{AB}(\psi^{AB}) \equiv \eta^{AB} D^{AB}(\psi^{AB}) \left[1 + 2\langle v_2^{DiJetA}\rangle\langle v_2^{DiJetB}\rangle \cos(2\psi^{AB})\right] \tag{84}$$

We have now absorbed the awkward normalization constant η^{AB} into the definition of $E^{AB}()$ — notice that η^{AB} does not appear in Eq. 83, thankfully. Close examination of Eq. 83 reveals that in order for the the joint distribution to have a total integral of 1, the function $E^{AB}()$ must be normalized to have a total integral of 1. This, then, is what fixes the value of η^{AB} as whatever is required to have Eq. 84 normalized to 1.

The upshot of all this is that we will not be able to access the function $D^{AB}()$ directly from the correlation function, but instead only its close cousin $E^{AB}()$. But that's OK, since D^{AB} and $E^{AB}()$ are nearly identical in shape as long as both are narrow peaks.

With all that said, we can now recognize that the integral in Eq. 83 is just the convolution of the function E^{AB} together with the function $J^A \circ J^B$, evaluated at $(\phi_A - \phi_B) - \pi$, and so we finally have the full joint distribution

$$\frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\mathrm{Di-Jet}}}{d\phi_A d\phi_B} = \frac{n^{AB}_{\mathrm{Di-Jet}}}{2\pi} J^A \circ J^B \circ E^{AB}((\phi_A - \phi_B) - \pi) \tag{85}$$

The form of Eq. 85 is, at last, just what you would have expected intiutively: it's a single peak centered on $\phi_A - \phi_B = \pi$, and its shape is the convolution of the individual jet fragmentations together with the acoplanarity. The extent to which jets "feel" the reaction plane enters only through the very small difference between $D^{AB}()$ and $E^{AB}()$, as in Eq. 84, so these effects will — as you would expect — have almost no influence on the di-jet peak in the correlation function. We can also see clearly now that the joint distribution in Eq. 85 does indeed integrate[10] to exactly $n^{AB}_{\mathrm{Di-Jet}}$, so all around it looks like a sensible result.

Now that the work of getting the joint distribution is done, the term in Eq. 61 for the Di-Jet pair type follows straightforwardly; with Eq. 85 we have

$$\begin{aligned} & \frac{2\pi}{n^A n^B} \int_0^{2\pi} d\phi_A \int_0^{2\pi} d\phi_B \, \delta(\Delta\phi - (\phi_A - \phi_B)) \, \frac{1}{N_{evt}} \frac{d^2 N^{AB}_{\mathrm{Di-Jet}}}{d\phi_A d\phi_B} \\ &= \frac{2\pi \, n^{AB}_{\mathrm{Di-Jet}}}{n^A n^B} J^A \circ J^B \circ E^{AB}(\Delta\phi - \pi) \end{aligned} \tag{86}$$

3.10 Summing It Up for the Two-Source Model

We can now realize Eq. 61 by summing up the terms for the different pair types: Eq. 62 for Flow-Flow pairs; Eq. 65 for Flow-Jet pairs; Eq. 68 for Unrelated-Jet pairs; Eq. 74 for Jet-Same-Side pairs; and Eq. 86 for Di-Jet pairs. So Eq. 61 now becomes

$$\begin{aligned} C(\Delta\phi) &= \frac{n^A_{\mathrm{Flow}} \, n^B_{\mathrm{Flow}}}{n^A \, n^B} \left[1 + 2 v_2^{FlowA} v_2^{FlowB} \cos(2\Delta\phi)\right] \\ & + \frac{n^A_{\mathrm{Flow}} \, n^B_{\mathrm{Jet}}}{n^A \, n^B} \left[1 + 2 v_2^{FlowA} j_2^B \langle v_2^{JetB} \rangle \cos(2\Delta\phi)\right] \end{aligned}$$

[10] The property of convolutions that we used before, that the integral of a convolution is just the product of the integrals of the original functions, also extends to multiple convolutions. Here, since we know that $J^A()$, $J^B()$ and $E^{AB}()$ are all normalized to 1 by definition, we know that the three-fold convolution is also normalized to 1.

$$
\begin{aligned}
&+ \frac{n^A_{\rm Jet}\, n^B_{\rm Flow}}{n^A\, n^B} \left[1 + 2 j^A_2 \langle v_2^{JetA}\rangle v_2^{FlowB} \cos(2\Delta\phi)\right] \\
&+ \frac{n^A_{\rm Jet}\, n^B_{\rm Jet}}{n^A\, n^B} \left[1 + 2\, j^A_2 \langle v_2^{JetA}\rangle\, j^B_2 \langle v_2^{JetB}\rangle \cos(2\Delta\phi)\right] \\
&+ \frac{2\pi\, n^{AB}_{\rm JetSameSide}}{n^A\, n^B} J^A \circ J^B(\Delta\phi) \;+\; \frac{2\pi\, n^{AB}_{\rm Di-Jet}}{n^A\, n^B} J^A \circ J^B \circ E^{AB}(\Delta\phi - \pi)
\end{aligned}
\tag{87}
$$

We can see that the first four lines of Eq. 87 are very similar, and each has a constant piece and a cosine piece. If we collect the constant pieces and cosine pieces together separately we get the suggestive result

$$
\begin{aligned}
C(\Delta\phi) \;=\;& \frac{n^A_{\rm Flow} n^B_{\rm Flow} + n^A_{\rm Flow} n^B_{\rm Jet} + n^A_{\rm Jet} n^B_{\rm Flow} + n^A_{\rm Jet} n^B_{\rm Jet}}{n^A\, n^B} \\
&+ 2 \left[\frac{n^A_{\rm Flow}\, n^B_{\rm Flow}}{n^A\, n^B} v_2^{FlowA} v_2^{FlowB} + \frac{n^A_{\rm Flow}\, n^B_{\rm Jet}}{n^A\, n^B} v_2^{FlowA} j^B_2 \langle v_2^{JetB}\rangle + \right. \\
&\left. \quad \frac{n^A_{\rm Jet}\, n^B_{\rm Flow}}{n^A\, n^B} j^A_2 \langle v_2^{JetA}\rangle v_2^{FlowB} + \frac{n^A_{\rm Jet}\, n^B_{\rm Jet}}{n^A\, n^B} j^A_2 \langle v_2^{JetA}\rangle j^B_2 \langle v_2^{JetB}\rangle \right] \cos(2\Delta\phi) \\
&+ \frac{2\pi\, n^{AB}_{\rm JetSameSide}}{n^A\, n^B} J^A \circ J^B(\Delta\phi) \;+\; \frac{2\pi\, n^{AB}_{\rm Di-Jet}}{n^A\, n^B} J^A \circ J^B \circ E^{AB}(\Delta\phi - \pi)
\end{aligned}
\tag{88}
$$

We can see that the first line in Eq. 88 is exactly equal to 1 by the definition of the n's. Careful examination of the sum in brackets on the second and third lines will show that it is equal to exactly the product $V_2^A\, V_2^B$, where V_2^A and V_2^B were defined earlier in Eq. 59 as the quadrupole strengths of the A and B full singles distributions, summed over both sources, with respect to Φ_{RP}. With this in hand, the correlation function takes on a very simple final form:

$$
\begin{aligned}
C(\Delta\phi) \;=\;& 1 + 2\, V_2^A\, V_2^B\, \cos(2\Delta\phi) \\
&+ \frac{2\pi\, n^{AB}_{\rm JetSameSide}}{n^A\, n^B} J^A \circ J^B(\Delta\phi) \;+\; \frac{2\pi\, n^{AB}_{\rm Di-Jet}}{n^A\, n^B} J^A \circ J^B \circ E^{AB}(\Delta\phi - \pi)
\end{aligned}
\tag{89}
$$

We can now – finally! – say that in the two-source model we would predict the angular correlation function to be the sum of a constant 1, plus a "standard" quadrupole term $2V_2^A V_2^B \cos(2\Delta\phi)$, plus two jet-fragmention peaks, one centered at $\Delta\phi = 0$ and the other at $\Delta\phi = \pi$.

We can make a quick double-check of the result in Eq. 89 by calculating the integral of the correlation function over $\Delta\phi$ and comparing the result to sum rule we derived in Sec. 2.2. From Eq. 89 we immediately find

$$
\int_0^{2\pi} d(\Delta\phi)\, C(\Delta\phi) \;=\; 2\pi \;+\; \frac{2\pi\, n^{AB}_{\rm JetSameSide}}{n^A\, n^B} \;+\; \frac{2\pi\, n^{AB}_{\rm Di-Jet}}{n^A\, n^B}
$$

$$= \quad 2\pi \left(\frac{n^A\, n^B \;+\; n^{AB}_{\rm JetSameSide} \;+\; n^{AB}_{\rm Di-Jet}}{n^A\, n^B} \right) \tag{90}$$

This is exactly what we would have expected, based on the sum rule derived in Eq. 33: it's the full rate of pairs per event, divided by the product of the singles rates per event, multiplied by 2π.

Looking at the form in Eq. 89 we can also make the following observations immediately:

• The strength of the quadrupole term will be the product of the quadrupole strengths of the A and B full singles distributions.

• The integrals of the jet-fragment peak terms will be $2\pi\, n^{AB}_{\rm JetSameSide}/n^A n^B$ on the near side and $2\pi\, n^{AB}_{\rm Di-Jet}/n^A n^B$ on the away side. Thus if these terms could be isolated, with their correct absolute normalization, it would be possible to extract the absolute number of jet fragment pairs and di-jet fragment pairs, since the quantities n^A and n^B are easy to measure separately.

• Note also that if the entire correlation function can be correctly absolutely normalized, then the jet-pair multiplicities can also be accessed without separating out the peak terms. The integral of the near-side half of the correlation function, *ie* in the range $-\pi/2 < \Delta\phi < \pi/2$, should yield $2\pi(1/2 \;+\; n^{AB}_{\rm JetSameSide}/n^A n^B)$. And, of course, the integral over the opposite-side half, $\pi/2 < \Delta\phi < 3\pi/2$, should yield $2\pi(1/2 \;+\; n^{AB}_{\rm Di-Jet}/n^A n^B)$. The strength of the quadrupole oscillation is irrelevant, since the integral over the $\cos(2\Delta\phi)$ will vanish for any interval of width π. Of course, the possible presence of other sources of correlation will have to be appreciated before any final result can be derived based on the integral of the correlation function.

3.11 Two-Source Model Result For Conditional Multiplicity

Once we have the relative-angle correlation function, it is an easy matter to predict the relative-angle conditional average multiplicity. Keeping the labels that A is the "Trigger" and B the "Companion", then inserting Eq. 89 into Eq. 49 we have

$$\begin{aligned} n^{B|A}(\Delta\phi) = \frac{1}{N^A}\,\frac{dN^{AB}}{d(\Delta\phi)} \quad &= \quad \frac{n^B}{2\pi} + \frac{n^B\, V_2^A\, V_2^B}{\pi}\;\cos(2\Delta\phi) \\ &\quad + \frac{n^{AB}_{\rm JetSameSide}}{n^A} J^A \circ J^B(\Delta\phi) \\ &\quad + \frac{n^{AB}_{\rm Di-Jet}}{n^A} J^A \circ J^B \circ E^{AB}(\Delta\phi - \pi) \end{aligned} \tag{91}$$

We can also double-check this result against the sum rule derived in Sec. 2.6. Integrating Eq. 91 over $\Delta\phi$ we have just

$$\int_0^{2\pi} d(\Delta\phi)\, n^{B|A}(\Delta\phi) = \int_0^{2\pi} d(\Delta\phi)\,\frac{1}{N^A}\,\frac{dN^{AB}}{d(\Delta\phi)} \quad = \quad \frac{n^A\, n^B \;+\; n^{AB}_{\rm JetSameSide} \;+\; n^{AB}_{\rm Di-Jet}}{n^A}$$

$$= \frac{N^A\, N^B \;+\; N^{AB}_{\rm JetSameSide} \;+\; N^{AB}_{\rm Di-Jet}}{N^A} \tag{92}$$

As with the correlation function sum rule check, this is exactly what we would expect for the conditional multiplicity based on Eq. 52: just the total number of AB pairs divided by the total number of A singles.

The observations we can make about the form in Eq. 91 for the relative-angle conditional average multiplicity are similar to those we made for the correlation function: it should be the sum of a constant term, a quadrupole term, and two jet-peak terms. Examination either of the jet-peak terms themselves, or of the integral with the constant term subtracted, can directly yield information on the quantities $n^{AB}_{\rm JetSameSide}/n^A$ and $n^{AB}_{\rm Di-Jet}/n^A$. As with the correlation function these can then be used to extract the rates for pair production from the jet and di-jet sources; or the ratio between the pairs and singles rates may be of interest by itself.

4 Glossary of Mathematical Notation